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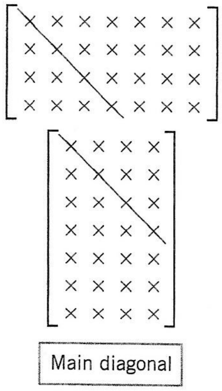
Singular Value Decomposition

Before turning to the main result in this section, we will find it useful to extend the notion of a "main diagonal" to matrices that are not square. We define the ***main diagonal*** of an $m \times n$ matrix to be the line of entries shown in Figure 9.5.1—it starts at the upper left corner and extends diagonally as far as it can go. We will refer to the entries on the main diagonal as the ***diagonal entries***.



▲ **Figure 9.5.1**

We are now ready to consider the main result in this section, which is concerned with a specific way of factoring a general $m \times n$ matrix A. This factorization, called ***singular value decomposition*** (abbreviated SVD) will be given in two forms, a brief form that captures the main idea, and an expanded form that spells out the details. The proof is given at the end of this section.

THEOREM 9.5.3 Singular Value Decomposition

If A is an $m \times n$ matrix, then A can be expressed in the form

$$A = U\Sigma V^T$$

where U and V are orthogonal matrices and Σ is an $m \times n$ matrix whose diagonal entries are the singular values of A and whose other entries are zero.

Harry Bateman (1882–1946)

Historical Note The term *singular value* is apparently due to the British-born mathematician Harry Bateman, who used it in a research paper published in 1908. Bateman emigrated to the United States in 1910, teaching at Bryn Mawr College, Johns Hopkins University, and finally at the California Institute of Technology. Interestingly, he was awarded his Ph.D. in 1913 by Johns Hopkins at which point in time he was already an eminent mathematician with 60 publications to his name.

[*Image: Courtesy of the Archives, California Institute of Technology*]

THEOREM 9.5.4 Singular Value Decomposition (Expanded Form)

If A is an $m \times n$ matrix of rank k, then A can be factored as

$$A = U\Sigma V^T = [\mathbf{u}_1 \ \ \mathbf{u}_2 \ \cdots \ \mathbf{u}_k \mid \mathbf{u}_{k+1} \ \cdots \ \mathbf{u}_m]\left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & 0_{k\times(n-k)} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \sigma_k & \\ \hline & 0_{(m-k)\times k} & & & 0_{(m-k)\times(n-k)} \end{array}\right]\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_k^T \\ \hline \mathbf{v}_{k+1}^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix}$$

in which U, Σ, and V have sizes $m \times m$, $m \times n$, and $n \times n$, respectively, and in which

(a) $V = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]$ *orthogonally diagonalizes A^TA.*

(b) *The nonzero diagonal entries of Σ are $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}, \ldots, \sigma_k = \sqrt{\lambda_k}$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the nonzero eigenvalues of A^TA corresponding to the column vectors of V.*

(c) *The column vectors of V are ordered so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$.*

(d) $\mathbf{u}_i = \dfrac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|} = \dfrac{1}{\sigma_i}A\mathbf{v}_i \quad (i = 1, 2, \ldots, k)$

(e) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ *is an orthonormal basis for* $\text{col}(A)$.

(f) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\}$ *is an extension of* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ *to an orthonormal basis for R^m.*

The vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are called the ***left singular vectors*** of A, and the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are called the ***right singular vectors*** of A.

Application: Data Compression

Tiger picture: Show it from webpage of Someone.

Original photo is Stored in a matrix

Data stor

$$A_{500 \times 800} \Leftrightarrow 400{,}000 \text{ KB}$$

We want to transmit it compressed as much as we can.

Using only the first 50 eigenvalues of A^TA or equivalently the first 50 Singular values of A, we obtain

$$A_{500 \times 800} = U_{500 \times 500} \, \Sigma_{500 \times 800} \, V^T_{800 \times 800}$$

$$\simeq \tilde{A}_{500 \times 800} = U_{500 \times 50} \, \Sigma_{50 \times 50} \, V^T_{50 \times 800}$$

Data Stored

Original Picture	SVD Simplified to 50 Sing. Values
$= 400 \text{ KB}$	$= 25{,}000 + 50 + 40{,}000$ $= 25 \text{ kB} + 0.05 \text{ kB} + 40 \text{ kB}$ $= 65.05 \text{ KB}$

Compare!